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Semilinear degenerate evolution inequalities with singular potential constructed from the generalized Greiner vector fields

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Abstract

We study the existence and nonexistence of global solutions to the degenerate evolution inequalities with singular potential constructed from the generalized Greiner vector fields. For the proof of the existence results, we use the method of supersolution and the modified Bessel function. The nonexistence results are established by the test function method.

MSC: 35H20; 35K65; 35G25

Keywords: Generalized Greiner vector fields; Degenerate inequality; Potential; Existence; Nonexistence

1. Introduction

Let

$$X_j = \frac{\partial}{\partial x_j} + 2d y_j |z|^{2d-2} \frac{\partial}{\partial l}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2d x_j |z|^{2d-2} \frac{\partial}{\partial l}, \tag{1.1}$$

with $j = 1, \ldots, n$, $x, y \in \mathbb{R}^n$, $l \in \mathbb{R}$, $z = x + \sqrt{-1}y$, $|z| = [\sum_{j=1}^n (x_j^2 + y_j^2)]^{\frac{1}{2}}$, $d \geq 1$, be the generalized Greiner vector fields. The generalized Greiner operator is defined as $\Delta_L = \sum_{j=1}^n \left(X_j^2 + Y_j^2\right)$. When $d = 1$, Δ_L becomes the sub-Laplacian Δ_{H^n} on the Heisenberg group H^n; see Folland [1]. If $d = 2, 3, \ldots$, Δ_L is the Greiner operator; see [2]. As is well known, the vector fields $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ in (1.1) do not possess left translation invariance for $d > 1$ and, if $d \neq 1, 2, 3, \ldots$, they do not meet the Hörmander condition [3].

We study the existence of global solutions to the degenerate parabolic inequality with singular potential

$$\frac{\partial u}{\partial t} - \Delta_L u + \lambda \frac{\psi}{\rho^2} u \geq |u|^q \tag{1.2}$$

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in two classes of subdomains of the half-space $\mathbb{S} := \mathbb{R}^{2n+1} \times (0, +\infty)$, for suitable $\lambda \geq 0$. Here $\rho = \rho(z, l)$ is a distance function (see (1.6) below), $\psi = \frac{|z|^{4d-2}}{\rho^{4d-2}}$. We also establish nonexistence results for higher-order degenerate evolution inequalities of the form

$$\frac{\partial^k u}{\partial t^k} - \Delta_L u + \lambda \frac{\psi}{\rho^2} u \geq |u|^q \tag{1.3}$$

in $\mathbb{S}$, where $\lambda \geq -\left(\frac{Q-2}{2}\right)^2$.

Since Fujita's famous classical paper [4], the studies of existence and nonexistence of global solutions to nonlinear heat equations on the Euclidean space and on nilpotent Lie groups have attracted much interest over the past few years; see [5–9] and the references therein. Levine and Meier [6] and Pascucci [7] obtained some sharp critical exponents for the reaction–diffusion equation on the Euclidean space and on nilpotent Lie groups, respectively. For the Euclidean case, Laptev [8] studied the nonexistence of global (nontrivial) solutions of some semilinear higher-order evolution inequalities. Hamidi and Laptev [9] proved nonexistence results for semilinear higher-order evolution inequalities with critical potential

$$\begin{cases} \dfrac{\partial^k u}{\partial t^k} - \Delta u + \dfrac{\lambda}{|x|^2} u \geq |u|^q, & (x,t) \in \mathbb{R}^n \times (0, +\infty),\ \lambda \geq -\left(\dfrac{n-2}{2}\right)^2,\ k \geq 1, \\ \dfrac{\partial^{k-1} u}{\partial t^{k-1}}(x, 0) \geq 0, & x \in \mathbb{R}^n, n \geq 3. \end{cases}$$

They found that, for some $q^*(n, \lambda, k)$, the above problem has no nontrivial global solution when $1 < q \leq q^*$. Hamidi and Laptev also established the existence of positive solutions to the parabolic inequality

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\lambda}{|x|^2} u \geq |u|^q, \quad \lambda \geq 0$$

in $\mathbb{R}^n \times (0, +\infty)$. In [9], the authors used the fact that the potential $\frac{\lambda}{|x|^2}$ is radial on Euclidean space. Naturally one wants to know whether the results in [9] can be generalized to inequalities with nonradial potential function or to degenerate inequalities.

The heat equation associated with the generalized Greiner vector fields is one of the important degenerate equations. In this present paper we study the existence and nonexistence results for semilinear degenerate evolution inequalities formed from the generalized Greiner vector fields. In the following we describe some known facts (see [10,11]) about the operator Δ_L and the family of vector fields $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$.

Denote the generalized gradient by $\nabla_L = (X_1, \ldots, X_n, Y_1, \ldots, Y_n)$. A natural family of anisotropic dilations attached to (1.1) is

$$\delta_a(z, l) = (az, a^{2d} l), \quad a > 0, (z, l) \in \mathbb{R}^{2n+1}. \tag{1.4}$$

It is easy to verify that

$$\mathrm{d}\delta_a(z, l) = a^Q \mathrm{d}z\mathrm{d}l, \tag{1.5}$$

where $Q := 2n + 2d$ is the homogeneous dimension related to (1.4) and $\mathrm{d}z\mathrm{d}l$ denotes the Lebesgue measure on $\mathbb{R}^{2n+1}$. The distance function is defined by

$$\rho(z, l) = (|z|^{4d} + l^2)^{\frac{1}{4d}}. \tag{1.6}$$

The next remark concerns the action of Δ_L on radial functions $u \in C^2$ depending only on $\rho(z, l)$. It is easy to show that

$$\Delta_L u(\rho) = \psi \left[u''(\rho) + \frac{Q-1}{\rho} u'(\rho) \right], \tag{1.7}$$

where $\psi := |\nabla_L \rho|^2 = \frac{|z|^{4d-2}}{\rho^{4d-2}}$. Clearly $0 \le \psi \le 1$. Furthermore, for functions $u \in C^2$ depending only on $|z|$, we have

$$\Delta_L u(|z|) = u''(|z|) + \frac{Q-2d-1}{|z|} u'(|z|). \tag{1.8}$$

We note that the potential $V = \lambda \frac{\psi}{\rho^2}$ in (1.2) and (1.3) is homogeneous of degree 2 with respect to the dilations δ_a and it has a weaker singularity than the inverse square function in the Euclidean case.

The plan of the paper is as follows. In Section 2 we recall a recursion formula concerning the modified Bessel function, define precisely the weak solution of problem (1.3) and establish some estimates which we shall use in the sequel. Section 3 is devoted to the existence results for (1.2). In Section 4, we prove the nonexistence of global solutions for higher-order degenerate evolution inequalities and systems.

2. Preliminaries and auxiliary estimates

Throughout this paper, the letter C denotes a positive constant which may vary from line to line but is independent of the terms which will take part in any limit process. Let $I_\sigma(r)$ be the modified Bessel function of order σ; then the following recursion formula holds (see [12]):

$$r I'_\sigma = \sigma I_\sigma + r I_{\sigma+1}. \tag{2.1}$$

Definition 2.1. Let $u(z,l,t) \in C(\mathbb{R}^{2n+1} \times [0,+\infty))$ which has locally integrable traces $\frac{\partial^i u}{\partial t^i}(z,l,0)$, $i = 1, \ldots, k-1$, on the hyperplane $t = 0$. The function $u(z,l,t)$ is called a weak solution of (1.3) in $\mathbb{S}$ if for any non-negative test function $\phi(z,l,t)$ with compact support, such that $\frac{\partial^k \phi}{\partial t^k} \in C(\mathbb{R}^{2n+1} \times [0,+\infty))$, $-\Delta_L \phi + \lambda \frac{\psi}{\rho^2}\phi \in L_1(\mathbb{R}^{2n+1} \times (0,+\infty))$, the following inequality holds:

$$\begin{aligned}\iiint_{\mathbb{S}} u \left[(-1)^k \frac{\partial^k \phi}{\partial t^k} - \Delta_L \phi + \lambda \frac{\psi}{\rho^2}\phi\right] \mathrm{d}z\mathrm{d}l\mathrm{d}t \ge{}& \iint_{\mathbb{R}^{2n+1}} \frac{\partial^{k-1} u}{\partial t^{k-1}}(z,l,0)\phi(z,l,0)\mathrm{d}z\mathrm{d}l \\ &+ \sum_{i=1}^{k-1}(-1)^i \iint_{\mathbb{R}^{2n+1}} \frac{\partial^{k-1-i} u}{\partial t^{k-1-i}}(z,l,0)\frac{\partial^i \phi}{\partial t^i}(z,l,0)\mathrm{d}z\mathrm{d}l \\ &+ \iiint_{\mathbb{S}} |u|^q \phi \mathrm{d}z\mathrm{d}l\mathrm{d}t. \end{aligned} \tag{2.2}$$

The definition of weak solutions of (1.3) in any subdomain $\Omega \subset \mathbb{S}$ is analogous.

Define the parameters

$$s^* = \frac{Q-2}{2} + \sqrt{\left(\frac{Q-2}{2}\right)^2 + \lambda}, \qquad s_* = -\frac{Q-2}{2} + \sqrt{\left(\frac{Q-2}{2}\right)^2 + \lambda}. \tag{2.3}$$

Using (1.7) one easily sees that

$$\left(-\Delta_L + \lambda \frac{\psi}{\rho^2(z,l)}\right)\rho^{s_*}(z,l) = \left(-\Delta_L + \lambda \frac{\psi}{\rho^2(z,l)}\right)\rho^{-s^*}(z,l) = 0 \tag{2.4}$$

for any $(z,l) \in \mathbb{R}^{2n+1} \backslash \{(0,0)\}$. In the following we construct a test function and obtain some estimates.

Let $\varphi : [0,+\infty) \to [0,1]$ be a smooth function which equals 1 on the interval $[0,1]$ and 0 on the interval $[2,+\infty)$. Let

$$\Phi = \varphi^{kp_0}$$

for some $p_0 > 1$ and $k \in \mathbb{N}$. A direct computation shows that

$$|\Phi^{(j)}|^p \le \left|\sum_{i=1}^{j} C \frac{(kp_0)!}{(kp_0-i)!}\varphi^{kp_0-i}\right|^p \le C\Phi^{p-1} \quad \text{for any } 1 \le j \le k,\ 1 < p \le p_0. \tag{2.5}$$

Let $\theta > 0$ and $R > 1$; for the function $\Phi(\frac{t}{R^\theta})$ we have $\operatorname{supp}\left|\Phi\left(\frac{t}{R^\theta}\right)\right| = \left\{(z,l,t) \in \mathbb{S} : 0 \le t \le 2R^\theta\right\}$, $\operatorname{supp}\left|\frac{d^k\Phi(\frac{t}{R^\theta})}{\mathrm{d}t^k}\right| = \{(z,l,t) \in \mathbb{S} : R^\theta \le t \le 2R^\theta\}$ and

$$\int_{\operatorname{supp}\left|\frac{d^k\Phi\left(\frac{t}{R^\theta}\right)}{\mathrm{d}t^k}\right|} \left|\frac{d^k\Phi\left(\frac{t}{R^\theta}\right)}{\mathrm{d}t^k}\right|^p \frac{1}{\Phi^{p-1}\left(\frac{t}{R^\theta}\right)} \mathrm{d}t \le CR^{-\theta(kp-1)}. \tag{2.6}$$

Set

$$\psi_R(z,l) := \Psi_R(\rho(z,l)) := \rho^s(z,l)\Phi\left(\frac{\rho(z,l)}{R}\right), \tag{2.7}$$

where the parameter s will be chosen later. It follows from (2.5) that

$$\left|\Psi_R'(\rho)\right|^p \le C\Phi^{p-1}\left(\frac{\rho}{R}\right)\rho^{(s-1)p}\left(1+\frac{\rho^p}{R^p}\right), \tag{2.8}$$

$$\left|\Psi_R''(\rho)\right|^p \le C\Phi^{p-1}\left(\frac{\rho}{R}\right)\rho^{(s-2)p}\left(1+\frac{\rho^p}{R^p}+\frac{\rho^{2p}}{R^{2p}}\right). \tag{2.9}$$

We then obtain for the operator $A := \Delta_L - \lambda\frac{\psi}{\rho^2}$:

$$\begin{aligned} |A\psi_R(z,l)|^p &= \psi^p \cdot \left|\Psi_R'' + \frac{Q-1}{\rho}\Psi_R' - \frac{\lambda}{\rho^2}\Psi_R\right|^p \\ &\le C\psi_R^{p-1}(z,l)\rho^{s-2p}\left(1+\frac{\rho^p}{R^p}+\frac{\rho^{2p}}{R^{2p}}\right). \end{aligned} \tag{2.10}$$

If $\lambda \ge 0$, we set $s = s_* \ge 0$. Since $A(\rho^{s_*}) = 0$ (from (2.4)), we then have $A\psi_R = 0$ for $\rho \le R$ and $\operatorname{supp}|A\psi_R| \subset \left\{(z,l) \in \mathbb{R}^{2n+1} : R \le \rho(z,l) \le 2R\right\}$. On the other hand, $1 + \frac{\rho^p}{R^p} + \frac{\rho^{2p}}{R^{2p}} \le C$ for some $C > 0$ when $R \le \rho \le 2R$. Therefore, $|A\psi_R(z,l)|^p \le C\psi_R^{p-1}(z,l)\rho^{s_*-2p}$ for $R \le \rho \le 2R$, which gives

$$\begin{aligned} \iint_{\operatorname{supp}|A\psi_R|} \frac{|A\psi_R(z,l)|^p}{\psi_R^{p-1}(z,l)}\mathrm{d}z\mathrm{d}l &\le C\iint_{\{(z,l)\in\mathbb{R}^{2n+1}:R\le\rho(z,l)\le 2R\}} \rho^{s_*-2p}\mathrm{d}z\mathrm{d}l \\ &\le CR^{s_*-2p+Q}. \end{aligned} \tag{2.11}$$

Define the function

$$\varphi_R(z,l,t) := \Phi\left(\frac{t}{R^\theta}\right)\psi_R(z,l). \tag{2.12}$$

It yields by (2.11),

$$\begin{aligned} \iiint_{\operatorname{supp}|A\varphi_R|} \frac{|A\varphi_R(z,l,t)|^p}{\varphi_R^{p-1}(z,l,t)}\mathrm{d}z\mathrm{d}l\mathrm{d}t &= \int_0^{2R^\theta}\Phi\left(\frac{t}{R^\theta}\right)\mathrm{d}t \iint_{\operatorname{supp}|A\psi_R|}\frac{|A\psi_R(z,l)|^p}{\psi_R^{p-1}(z,l)}\mathrm{d}z\mathrm{d}l \\ &\le CR^{\theta+s_*-2p+Q}. \end{aligned} \tag{2.13}$$

Analogously, using (2.6), we get

$$\iiint_{\operatorname{supp}\left|\frac{\partial^k\varphi_R}{\partial t^k}\right|}\left|\frac{\partial^k\varphi_R(z,l,t)}{\partial t^k}\right|^p\frac{1}{\varphi_R^{p-1}(z,l,t)}\mathrm{d}z\mathrm{d}l\mathrm{d}t$$

$$=\int_{\operatorname{supp}\left|\frac{d^k\Phi\left(\frac{t}{R^\theta}\right)}{\mathrm{d}t^k}\right|}\left|\frac{\mathrm{d}^k\Phi\left(\frac{t}{R^\theta}\right)}{\mathrm{d}t^k}\right|^p\frac{1}{\Phi^{p-1}\left(\frac{t}{R^\theta}\right)}\mathrm{d}t\iint_{\{(z,l)\in\mathbb{R}^{2n+1}:\rho(z,l)<2R\}}\psi_R(z,l)\mathrm{d}z\mathrm{d}l$$

$$\le CR^{s_*+Q-\theta(kp-1)}. \tag{2.14}$$

When $\theta=\frac{2}{k}$, the power in (2.13) equals the one in (2.14): $\theta+s_*-2p+Q=s_*+Q-\theta(kp-1)=s_*-2p+Q+\frac{2}{k}$. Thus, we eventually arrive at

$$J_p:=\iiint_{\operatorname{supp}|(-1)^k\frac{\partial^k\varphi_R}{\partial t^k}-A\varphi_R|}\frac{\left|(-1)^k\frac{\partial^k\varphi_R}{\partial t^k}-A\varphi_R\right|^p}{\varphi_R^{p-1}}\mathrm{d}z\mathrm{d}l\mathrm{d}t$$

$$\le CR^{s_*-2p+Q+\frac{2}{k}}. \tag{2.15}$$

If $-(\frac{Q-2}{2})^2\le\lambda<0$, we then take $s=-s^*\le 0$ which gives $A\psi_R=0$ for $\rho\le R$ and $\operatorname{supp}|A\psi_R|\subset\{(z,l)\in\mathbb{R}^{2n+1}:R\le\rho\le 2R\}$. Similar to the estimates (2.11) and (2.15), we have

$$\iint_{\operatorname{supp}|A\psi_R|}\frac{|A\psi_R(z,l)|^p}{\psi_R^{p-1}(z,l)}\mathrm{d}z\mathrm{d}l\le CR^{-s^*-2p+Q} \tag{2.16}$$

and

$$J_p:=\iiint_{\operatorname{supp}|(-1)^k\frac{\partial^k\varphi_R}{\partial t^k}-A\varphi_R|}\frac{\left|(-1)^k\frac{\partial^k\varphi_R}{\partial t^k}-A\varphi_R\right|^p}{\varphi_R^{p-1}}\mathrm{d}z\mathrm{d}l\mathrm{d}t\le CR^{-s^*-2p+Q+\frac{2}{k}}. \tag{2.17}$$

3. Existence results

If $\Omega\subset\mathbb{R}^{2n+1}$ is a domain with the property that there exists a positive constant $M>0$ such that $|z|\le M$ for every $(z,l)\in\Omega$, then we denote Ω by Ω_M. It is easy to see that all bounded domains in $\mathbb{R}^{2n+1}$ satisfy the property for suitable constants $M>0$.

Consider the parabolic problem

$$\begin{cases}\dfrac{\partial u}{\partial t}-\Delta_L u+\lambda\dfrac{\psi}{\rho^2}u\ge|u|^q, & (z,l,t)\in\Sigma_M\times(0,+\infty),\\ u(z,l,0)=u_0(z,l)\ge 0, & (z,l)\in\Sigma_M\end{cases} \tag{3.1}$$

with $\Sigma_M:=\{(z,l)\in\Omega_M:0\le\frac{\lambda l^2}{|z|^{4d}+l^2}\le\sqrt{\lambda+(\frac{Q-2}{2})^2}\}\subset\Omega_M$ and λ be such that: $\sqrt{\lambda+(\frac{Q-2}{2})^2}\ge\frac{d}{2d-1}(M^2+Q-2)$.

Theorem 3.1. *If $q>q^*\equiv 1+\frac{2}{s^*+2}$, then the problem (3.1) has nontrivial global solutions.*

Proof. Let $v(z,l,t)$ be a positive solution of

$$\begin{cases}\dfrac{\partial v}{\partial t}-\Delta_L v+\lambda\dfrac{\psi}{\rho^2}v\ge 0, & (z,l,t)\in\Sigma_M\times(0,+\infty),\\ v(z,l,0)=v_0(z,l)\ge 0, & (z,l)\in\Sigma_M,\end{cases} \tag{3.2}$$

and let

$$w(z,l,t):=\alpha(t)v(z,l,t).$$

Then w will be a solution of (3.1) provided

$$\alpha'(t) = [\alpha(t)]^q \|v(\cdot,\cdot,t)\|^{q-1}_{L^\infty(\Sigma_M)}, \quad t > 0 \tag{3.3}$$

with $\alpha(0) = \alpha_0 > 0$. The solution of (3.3) will be global in t if, and only if,

$$\int_0^{+\infty} \|v(\cdot,\cdot,t)\|^{q-1}_{L^\infty(\Sigma_M)} \,\mathrm{d}t < +\infty \tag{3.4}$$

and

$$0 < \alpha_0 < \left[(q-1)\int_0^{+\infty} \|v(\cdot,\cdot,t)\|^{q-1}_{L^\infty(\Sigma_M)} \,\mathrm{d}t\right]^{-\frac{1}{q-1}}.$$

Thus, it remains to construct $v(z,l,t)$ which satisfies (3.4). We define the nonradial function

$$v(z,l,t) = \frac{1}{t+1}\frac{1}{|z|^{\frac{1}{2}(Q-2)}} I_\sigma\left(\frac{|z|}{2(t+1)}\right)\exp\left(-\frac{|z|^2+1}{4(t+1)}\right),$$

where $\sigma := s_* + \frac{Q-2}{2} = \sqrt{\lambda + \left(\frac{Q-2}{2}\right)^2}$ and I_σ denotes the modified Bessel function of order σ [12]. Using (1.8) and (2.1) we arrive at for the operator $P := \frac{\partial}{\partial t} - \Delta_L + \lambda\frac{\psi}{\rho^2}$ and the function v,

$$\begin{aligned}
Pv(z,l,t) &= \aleph\left\{\left[\frac{(Q-2)(Q-4d-2)}{4} - \frac{d|z|^2}{t+1} + \frac{|z|^2}{4(t+1)^2} + \lambda\psi^{1+\frac{1}{2d-1}}\right]I_\sigma \right.\\
&\quad \left. + \frac{(2d-1)|z|}{2(t+1)}I'_\sigma - \frac{|z|^2}{4(t+1)^2}I''_\sigma\right\}\\
&= \aleph\left\{\left[-d(Q-2) - \frac{d|z|^2}{t+1} - \lambda\left(1-\psi^{1+\frac{1}{2d-1}}\right)\right]I_\sigma + \frac{d|z|}{t+1}I'_\sigma\right\}\\
&= \aleph\left\{\left[-d(Q-2) - \frac{d|z|^2}{t+1} - \lambda\left(1-\psi^{1+\frac{1}{2d-1}}\right)\right]I_\sigma + \frac{d|z|}{t+1}I_{\sigma+1}\right.\\
&\quad \left. + 2d\sqrt{\lambda + \left(\frac{Q-2}{2}\right)^2}I_\sigma\right\},
\end{aligned} \tag{3.5}$$

with $\aleph = \frac{1}{t+1}\frac{1}{|z|^{\frac{1}{2}(Q+2)}}\exp(-\frac{|z|^2+1}{4(t+1)}) \geq 0$. Since $\sqrt{\lambda + (\frac{Q-2}{2})^2} \geq \frac{d}{2d-1}(M^2 + Q - 2)$, it follows from (3.5) that $Pv(z,l,t) \geq 0$ in $\Sigma_M \times (0,+\infty)$. Recall the asymptotic behaviours of I_σ [12]:

$$I_\sigma(r) \approx \begin{cases} \dfrac{r^\sigma}{2^\sigma \Gamma(\sigma+1)}, & \text{as } r \to 0^+,\\[2mm] \dfrac{\mathrm{e}^r}{\sqrt{2\pi r}}, & \text{as } r \to +\infty. \end{cases} \tag{3.6}$$

Since $\|v(\cdot,\cdot,t)\|^{q-1}_{L^\infty(\Sigma_M)} = (t+1)^{-\frac{q-1}{q^*-1}}[(t+1)^{\frac{1}{q^*-1}}\|v(\cdot,\cdot,t)\|_{L^\infty(\Sigma_M)}]^{q-1}$ and $\int_0^{+\infty}(t+1)^{-\frac{q-1}{q^*-1}}\mathrm{d}t < +\infty$ for any $q > q^*$, in order to show that (3.4) holds for any $q > q^*$, it suffices to show that

$$\limsup_{t\to+\infty}(t+1)^{\frac{1}{q^*-1}}\|v(\cdot,\cdot,t)\|_{L^\infty(\Sigma_M)} < +\infty. \tag{3.7}$$

Now $v(z,l,t)$ vanishes at $|z| = 0, +\infty$ for each t. Thus there exists $0 < r^*(t) < +\infty$, such that

$$v\left(r^*(t),t\right) = \|v(\cdot,\cdot,t)\|_{L^\infty(\Sigma_M)}.$$

Let

$$\begin{aligned}\vartheta(t) &:= (t+1)^{\frac{1}{q^*-1}} v\left(r^*(t), t\right) \\ &= (t+1)^{\frac{s_*}{2}} \left(\frac{r^*(t)}{t+1}\right)^{-\frac{Q-2}{2}} I_\sigma\left(\frac{r^*(t)}{2(t+1)}\right) \mathrm{e}^{-\frac{(r^*(t))^2+1}{4(t+1)}}.\end{aligned}$$

Set $y^*(t) = \frac{r^*(t)}{2(t+1)}$; then

$$\vartheta(t) = 2^{-\frac{Q-2}{2}} \mathrm{e}^{-\frac{1}{4(t+1)}} (t+1)^{\frac{s_*}{2}} \left(y^*(t)\right)^{-\frac{Q-2}{2}} I_\sigma\left(y^*(t)\right) \mathrm{e}^{-(t+1)\left(y^*(t)\right)^2}.$$

Suppose that, on some sequence $\{t_k\}_{k\in\mathbb{N}}$, $\vartheta(t_k) \to +\infty$ as $t_k \to +\infty$. If, on some subsequence, $y^*(t_k) \to +\infty$, then

$$\vartheta(t_k) \approx C(t_k+1)^{\frac{s_*}{2}} \left(y^*(t_k)\right)^{-\frac{Q-1}{2}} \mathrm{e}^{y^*(t_k)} \mathrm{e}^{-(t_k+1)\left(y^*(t_k)\right)^2} \quad \text{as } t_k \to +\infty,$$

which implies that $\vartheta(t_k) \to 0$, as $t_k \to +\infty$, on such a subsequence. If, on the other hand, $y^*(t_k) \to 0$,

$$\vartheta(t_k) \approx C\left[(t_k+1)(y^*(t_k))^2\right]^{\frac{s_*}{2}} \mathrm{e}^{-(t_k+1)\left(y^*(t_k)\right)^2} \quad \text{as } t_k \to +\infty$$

from which we conclude that $\vartheta(t_k)$ is bounded, since $r^{\frac{s_*}{2}} \mathrm{e}^{-r}$ is bounded on $[0, +\infty)$. Therefore if $\vartheta(t_k) \to +\infty$, we must have two constants A and B such that the sequence $\{y^*(t_k)\}_{k\in\mathbb{N}}$ satisfies

$$0 < A \le y^*(t_k) \le B < +\infty.$$

In this case, the expression $\vartheta(t_k)$ is clearly bounded.

Therefore, there is no sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $\vartheta(t_k) \to +\infty$ as $t_k \to +\infty$. This ends the proof. □

Next, we extend the existence result to the following exterior problem of the parabolic inequality

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_L u + \lambda \dfrac{\psi}{\rho^2} u \ge |u|^q, & (z, l, t) \in D \times (0, T), \lambda \ge 0, T > 0, \\ u(z, l, 0) = \bar{u}_0(z, l) \ge 0, & (z, l) \in D, \end{cases} \tag{3.8}$$

where $D \subset \mathbb{R}^{2n+1}$ satisfies the condition that $\frac{(\rho(z,l)-1)^2}{4(T+1)} \ge 1 + \sqrt{\lambda + (\frac{Q-2}{2})^2}$ for every $(z, l) \in D$.

Theorem 3.2. *If $q > 1 + \frac{2}{s^*+2}$, then nontrivial global solutions of (3.8) exist.*

Proof. Let $\bar{v}(z, l, t)$ be a positive solution of $P\bar{v} \ge 0$ on $D \times [0, T]$, with $\bar{v}(z, l, 0) = \bar{v}_0(z, l) \ge 0$ on D. Let the function $\bar{w}$ be defined on $D \times [0, T)$ by $\bar{w}(z, l, t) = \bar{\alpha}(t)\bar{v}(z, l, t)$. Then $\bar{w}(z, l, t)$ will be a global solution of (3.8) provided

$$\bar{\alpha}'(t) = [\bar{\alpha}(t)]^q \, \|\bar{v}(\cdot, \cdot, t)\|_{L^\infty(D)}^{q-1}, \quad 0 \le t \le T \tag{3.9}$$

with $\bar{\alpha}(0) = \bar{\alpha}_0 > 0$. The solution of (3.9) will be global in t if, and only if,

$$\int_0^{+\infty} \|\bar{v}(\cdot, \cdot, t)\|_{L^\infty(D)}^{q-1} \, \mathrm{d}t < +\infty \tag{3.10}$$

and

$$0 < \bar{\alpha}_0 < \left[(q-1) \int_0^{+\infty} \|\bar{v}(\cdot, \cdot, t)\|_{L^\infty(D)}^{q-1} \, \mathrm{d}t\right]^{-\frac{1}{q-1}}.$$

Therefore, it remains to construct $\bar{v}(z, l, t)$ which satisfies (3.10). Consider the function

$$\bar{v}(z, l, t) = \frac{1}{t+1} \frac{1}{(\rho(z, l))^{\frac{1}{2}(Q-2)}} I_\sigma\left(\frac{\rho(z, l)}{2(t+1)}\right) \exp\left(-\frac{\rho^2(z, l)+1}{4(t+1)}\right),$$

where $\sigma := s_* + \frac{Q-2}{2}$. We see after using (1.7) and (2.1) that

$$\begin{aligned} P\bar{v}(z,l,t) &= \bar{\aleph}\left[-I_\sigma - \frac{\rho(z,l)}{2(t+1)} I'_\sigma + \frac{\rho^2(z,l)+1}{4(t+1)} I_\sigma\right] \\ &= \bar{\aleph}\left[-I_\sigma - \sqrt{\lambda + \left(\frac{Q-2}{2}\right)^2} I_\sigma - \frac{\rho(z,l)}{2(t+1)} I_{\sigma+1} + \frac{\rho^2(z,l)+1}{4(t+1)} I_\sigma\right] \\ &\geq \bar{\aleph}\left[-I_\sigma - \sqrt{\lambda + \left(\frac{Q-2}{2}\right)^2} I_\sigma + \frac{(\rho(z,l)-1)^2}{4(t+1)} I_\sigma\right] \\ &\geq 0, \end{aligned}$$

with $\bar{\aleph} = \frac{1}{(t+1)^2} \frac{1-\psi}{(\rho(z,l))^{\frac{1}{2}(Q-2)}} \exp(-\frac{\rho^2(z,l)+1}{4(t+1)})$, in $D \times [0,T]$. Hence, similar to the argument in the proof of Theorem 3.1, we conclude the result. □

4. Nonexistence results for higher-order inequalities (systems)

First we present the nonexistence of global solutions of the problem

$$\begin{cases} \dfrac{\partial^k u}{\partial t^k} - \Delta_L u + \lambda \dfrac{\psi}{\rho^2} u \geq |u|^q, & (z,l,t) \in \mathbb{S}, \lambda \geq -\left(\dfrac{Q-2}{2}\right)^2, k \geq 1, \\ \dfrac{\partial^{k-1} u}{\partial t^{k-1}}(z,l,0) \geq 0, & (z,l) \in \mathbb{R}^{2n+1}. \end{cases} \tag{4.1}$$

Theorem 4.1. *If one of the following conditions holds true:*

(h1) $\lambda \geq 0$ *and* $1 < q \leq q^* = 1 + \dfrac{2}{s^* + \frac{2}{k}}$,

(h2) $-\left(\dfrac{Q-2}{2}\right)^2 \leq \lambda < 0$ *and* $1 < q \leq q^* = 1 + \dfrac{2}{-s_* + \frac{2}{k}}$,

then the problem (4.1) has no nontrivial global solution.

Proof. Let $u(z,l,t)$ be a global nontrivial solution of (4.1). Substituting $\phi(z,l,t) = \varphi_R(z,l,t)$ as a test function in (2.2), where φ_R is as in (2.12), $p = q' > 1$, $s = s_*$ or $s = -s^*$, and $\theta = \frac{2}{k}$, and noting that $\frac{\partial^{k-1} u}{\partial t^{k-1}}(z,l,0) \geq 0$ and $\frac{\partial^i \varphi_R}{\partial t^i}(z,l,0) \equiv 0$, $i = 1, \ldots, k-1$, we get

$$\iiint_{\mathbb{S}} |u|^q \varphi_R \mathrm{d}z\mathrm{d}l\mathrm{d}t \leq \iiint_{\mathrm{supp}\left|(-1)^k \frac{\partial^k \varphi_R}{\partial t^k} - A\varphi_R\right|} u \left[(-1)^k \frac{\partial^k \varphi_R}{\partial t^k} - A\varphi_R\right] \mathrm{d}z\mathrm{d}l\mathrm{d}t. \tag{4.2}$$

Using Hölder inequality for the integral on the right hand side of (4.2) yields

$$\begin{aligned} &\iiint_{\mathrm{supp}\left|(-1)^k \frac{\partial^k \varphi_R}{\partial t^k} - A\varphi_R\right|} |u|^q \varphi_R \mathrm{d}z\mathrm{d}l\mathrm{d}t + \iiint_{\{(z,l,t)\in\mathbb{S}:\varphi_R = \rho^s(z,l)\}} |u|^q \rho^s(z,l) \mathrm{d}z\mathrm{d}l\mathrm{d}t \\ &\leq \left(\iiint_{\mathrm{supp}\left|(-1)^k \frac{\partial^k \varphi_R}{\partial t^k} - A\varphi_R\right|} |u|^q \varphi_R \mathrm{d}z\mathrm{d}l\mathrm{d}t\right)^{\frac{1}{q}} J_{q'}^{\frac{1}{q'}}, \end{aligned} \tag{4.3}$$

and then

$$\iiint_{\{(z,l,t)\in\mathbb{S}:\varphi_R = \rho^s(z,l)\}} |u|^q \rho^s(z,l) \mathrm{d}z\mathrm{d}l\mathrm{d}t \leq J_{q'}. \tag{4.4}$$

First case. If $\lambda \geq 0$, taking $s = s_*$, we get from (2.15) that

$$\iiint_{\{(z,l,t)\in\mathbb{S}:\varphi_R=\rho^{s_*}(z,l)\}} |u|^q \rho^{s_*}(z,l)\mathrm{d}z\mathrm{d}l\mathrm{d}t \leq J_{q'} \leq CR^{s_*-2q'+Q+\frac{2}{k}}. \tag{4.5}$$

Since in this case $1 < q \leq q^* = 1 + \frac{2}{s^*+\frac{2}{k}}$, i.e. $s_* - 2q' + Q + \frac{2}{k} \leq 0$, letting $R \to +\infty$ in (4.5) leads to

$$\iiint_{\mathbb{S}} |u|^q \rho^{s_*}(z,l)\mathrm{d}z\mathrm{d}l\mathrm{d}t \leq C.$$

On the other hand, from $\varphi_R(z,l,t) \leq \rho^{s_*}(z,l)$ and the general properties of the Lebesgue integral we get

$$\begin{aligned}\iiint_{\mathrm{supp}\left|(-1)^k\frac{\partial^k\varphi_R}{\partial t^k}-A\varphi_R\right|} |u|^q\varphi_R\mathrm{d}z\mathrm{d}l\mathrm{d}t &\leq \iiint_{\mathrm{supp}\left|(-1)^k\frac{\partial^k\varphi_R}{\partial t^k}-A\varphi_R\right|} |u|^q\rho^{s_*}\mathrm{d}z\mathrm{d}l\mathrm{d}t \\ &= \varepsilon(R) \to 0\end{aligned}$$

as $R \to +\infty$. Then (4.3) becomes

$$\iiint_{\{(z,l,t)\in\mathbb{S}:\varphi_R=\rho^{s_*}(z,l)\}} |u|^q \rho^{s_*}(z,l)\mathrm{d}z\mathrm{d}l\mathrm{d}t \leq \varepsilon^{\frac{1}{q}}(R)C^{\frac{1}{q'}} \to 0$$

as $R \to +\infty$. This implies $\iiint_{\mathbb{S}} |u|^q\rho^{s_*}(z,l)\mathrm{d}z\mathrm{d}l\mathrm{d}t = 0$. Therefore, the solution $u(z,l,t)$ must be trivial under the hypothesis (h1).

Second case. If $\lambda < 0$, then we can choose $s = -s^*$ and the estimate (2.17) gives

$$\iiint_{\{(z,l,t)\in\mathbb{S}:\varphi_R=\rho^{-s^*}(z,l)\}} |u|^q \rho^{-s^*}(z,l)\mathrm{d}z\mathrm{d}l\mathrm{d}t \leq J_{q'} \leq CR^{-s^*-2q'+Q+\frac{2}{k}}.$$

Similarly, the nonexistence of nontrivial solutions is deduced in the case $-s^* - 2q' + Q + \frac{2}{k} \leq 0$. □

The next result deals with the inhomogeneous problem

$$\begin{cases} \dfrac{\partial^k u}{\partial t^k} - \Delta_L u + \lambda\dfrac{\psi}{\rho^2}u \geq |u|^q + w(z,l), & (z,l,t) \in \mathbb{S}, \\ \dfrac{\partial^{k-1}u}{\partial t^{k-1}}(z,l,0) \geq 0, & (z,l) \in \mathbb{R}^{2n+1} \end{cases} \tag{4.6}$$

with $w(z,l) \in L^1_{\mathrm{loc}}(\mathbb{R}^{2n+1})$, $w(z,l) \geq 0$. The weak solution of problem (4.6) can be understood in the sense of Definition 2.1 with the extra term $\iiint_{\mathbb{S}} w(z,l)\phi\mathrm{d}z\mathrm{d}l\mathrm{d}t$.

Theorem 4.2. *If $\lambda \geq 0$ and $1 < q < 1 + \frac{2}{s^*}$, or $-(\frac{Q-2}{2})^2 \leq \lambda < 0$ and $1 < q < 1 + \frac{2}{-s_*}$, then the Problem (4.6) has no nontrivial global solution for any arbitrary small $w(z,l) \geq 0$, $w(z,l) \not\equiv 0$.*

The proof of Theorem 4.2 is similar to that of Theorem 4.1, so we omit it. Consider the following systems

$$\begin{cases} \dfrac{\partial^k u}{\partial t^k} - \Delta_L u + \lambda\dfrac{\psi}{\rho^2}u \geq |v|^{q_1}, & (z,l,t) \in \mathbb{S}, \\ \dfrac{\partial^k v}{\partial t^k} - \Delta_L v + \lambda\dfrac{\psi}{\rho^2}v \geq |u|^{q_2}, & (z,l,t) \in \mathbb{S}, \\ \dfrac{\partial^{k-1}u}{\partial t^{k-1}}(z,l,0) \geq 0,\ \dfrac{\partial^{k-1}v}{\partial t^{k-1}}(z,l,0) \geq 0, & (z,l) \in \mathbb{R}^{2n+1}. \end{cases} \tag{4.7}$$

Theorem 4.3. *Let $q_1, q_2 > 1$, $\gamma_1 = \frac{q_1+1}{q_1q_2-1}$ and $\gamma_2 = \frac{q_2+1}{q_1q_2-1}$. If $\lambda \geq 0$ and $\max\{\gamma_1,\gamma_2\} \geq \frac{s^*+\frac{2}{k}}{2}$, or $-(\frac{Q-2}{2})^2 \leq \lambda < 0$ and $\max\{\gamma_1,\gamma_2\} \geq \frac{-s_*+\frac{2}{k}}{2}$, then (4.7) has no nontrivial global solution.*

Proof. We only prove the case $\lambda \geq 0$, the other case being treated similarly. Using Hölder inequality, we deduce from Definition 2.1 that

$$\iiint_{\mathbb{S}} |v|^{q_1} \varphi_R \mathrm{d}z\mathrm{d}l\mathrm{d}t \leq \left(\iiint_{\mathrm{supp}\left|(-1)^k \frac{\partial^k \varphi_R}{\partial t^k} - A\varphi_R\right|} |u|^{q_2} \varphi_R \mathrm{d}z\mathrm{d}l\mathrm{d}t \right)^{\frac{1}{q_2}} J_{q_2'}^{\frac{1}{q_2'}}, \tag{4.8}$$

$$\iiint_{\mathbb{S}} |u|^{q_2} \varphi_R \mathrm{d}z\mathrm{d}l\mathrm{d}t \leq \left(\iiint_{\mathrm{supp}\left|(-1)^k \frac{\partial^k \varphi_R}{\partial t^k} - A\varphi_R\right|} |v|^{q_1} \varphi_R \mathrm{d}z\mathrm{d}l\mathrm{d}t \right)^{\frac{1}{q_1}} J_{q_1'}^{\frac{1}{q_1'}}. \tag{4.9}$$

Substituting (4.9) into (4.8) we get

$$\iiint_{\mathbb{S}} |v|^{q_1} \varphi_R \mathrm{d}z\mathrm{d}l\mathrm{d}t \leq \left(\iiint_{\mathrm{supp}\left|(-1)^k \frac{\partial^k \varphi_R}{\partial t^k} - A\varphi_R\right|} |v|^{q_1} \varphi_R \mathrm{d}z\mathrm{d}l\mathrm{d}t \right)^{\frac{1}{q_1 q_2}} J_{q_1'}^{\frac{1}{q_1' q_2}} J_{q_2'}^{\frac{1}{q_2'}}. \tag{4.10}$$

Applying (2.15) to (4.10) yields

$$\iiint_{\mathbb{S}} |v|^{q_1} \varphi_R \mathrm{d}z\mathrm{d}l\mathrm{d}t \leq \left(J_{q_1'}^{q_1 - 1} J_{q_2'}^{q_1(q_2 - 1)} \right)^{\frac{1}{q_1 q_2 - 1}} \leq C R^{s^* + \frac{2}{k} - 2\gamma_1}. \tag{4.11}$$

Therefore, we can conclude that $v(z, l, t)$ must be trivial under the hypothesis $\gamma_1 \geq \frac{s^* + \frac{2}{k}}{2}$, and so also must $u(z, l, t)$.

Analogously, substituting (4.8) into (4.9) gives

$$\iiint_{\mathbb{S}} |u|^{q_2} \varphi_R \mathrm{d}z\mathrm{d}l\mathrm{d}t \leq \left(J_{q_1'}^{q_2(q_1 - 1)} J_{q_2'}^{q_2 - 1} \right)^{\frac{1}{q_1 q_2 - 1}} \leq C R^{s^* + \frac{2}{k} - 2\gamma_2}. \tag{4.12}$$

Hence, there is no nontrivial $u(z, l, t)$ and nontrivial $v(z, l, t)$ for $\gamma_2 \geq \frac{s^* + \frac{2}{k}}{2}$. This ends the proof. □

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